On Remembering Cardano Anew

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AND TONY ROTHMAN

Years Ago *features essays by historians and mathematicians that take us back in time. Whether addressing special topics or general trends, individual mathematicians or "schools" (as in schools of fish), the idea is always the same: to shed new light on the mathematics of the past. Submissions are welcome.*

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to **David E. Rowe,** rowe@mathematik.uni-mainz.de

Editor's Introduction

The celestial spheres must have aligned in an unusual fashion last year, heralding the intellectual achievements of the Italian polymath Gerolamo Cardano (1501–1576).[1] How else to account for the near simultaneous appearance of two entirely independent, and yet unusually penetrating essays on this exotic figure from the Italian Renaissance? In what follows, Albrecht Heeffer and Tony Rothman approach their subject from entirely different directions and with quite distinct purposes in mind. And yet both manage to tell us something new and important about this legendary figure, whose life and exploits have long fascinated historians, mathematicians, but especially those with a sense for the dramatic. Whereas Heeffer focuses on one of Cardano's favorite mathematical motifs, Rothman takes apart some of the tall tales that have crept into more recent accounts of his storied life.

Some of Cardano's exploits as a mathematician are by now quite familiar. His work on probability, published posthumously in *Liber de Ludo Aleae* (*The Book on Games of Chance*), provided the main motivation for Oystein Ore's biography, *Cardano: The Gambling Scholar*. Still, mathematicians are most likely to connect Cardano with the well-known formula for solving cubic equations, a theory that received its first systematic treatment in his *Ars Magna* from 1545. This classic has often been paired with two others from the 1540s, the decade that saw the publication of Copernicus's *De revolutionibus orbium coelestium* and Vesalius's *De humani corporis fabrica*. Yet, as Heeffer shows, Cardano's intellectual interests went far beyond what we conceive of today as conventionally mathematical. Indeed, he was, like Kepler, an astrologer,[2] a trade practically synonymous with that of the *mathematicus* in this era. But Cardano was also a professional physician who, like Vesalius, broke new ground for medical practice in the Renaissance. Nancy Siraisi emphasized the novelty of his experimental work by drawing on his advice to the medical practitioner, who should "always have at hand a clock and a mirror": the first to keep track of time, the second to observe changes in the condition of his body.[3] One might say, Cardano turned the biblical proverb on its head to say "physician know thyself," and Siraisi relates this side of Cardano's scientific interests to his other pursuits, including astrology and autobiography.

As for his interest in the "Great Art" of solving algebraic equations, this led to one of the most famous of all disputes in the history of mathematics, the feud that ensued when Niccolò Tartaglia accused Cardano of having stolen his

[1]Known in France as Jérôme Cardan or, in the scholarly world of his day, under the Latin name Hieronymus Cardanus.

[2]Grafton, Anthony, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer*, Harvard University Press, 2001.

[3]Siraisi, Nancy G. *The Clock and the Mirror: Girolamo Cardano and Renaissance Medicine*. Princeton University Press, 1997.

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secret algorithm for solving the cubic, more precisely the special case: $x^3 + ax = b$. In his *Ars Magna*, Cardano took pains to spell out the earlier circumstances when he wrote: "Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of Venice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in response to my entreaties, though withholding the demonstration." This passage comes from Chapter 11 in the English translation by T. Richard Witmer, who elaborates on these surrounding circumstances and the famous dispute that ensued from them in his introduction.[4]

Rothman's essay takes up this "Great Feud" once again in order to show how the legends surrounding Cardano's life have grown over the years, aided and abetted more recently by the internet. This phenomenon is truly a pervasive one in the history of mathematics, though it is by no means new; Witmer pointed to the problematic account in Herbert W. Turnbull's *The Great Mathematicians* (4th ed., 1951), which was reproduced five years later in James R. Newman's *The World of Mathematics*.[5] Still, the fabrications Rothman writes of here are surely in a different category altogether. Since someone took the trouble to amend the Wikipedia article on Cardano, this particular flaw (alluded to below) has in the meantime been repaired, though of course many similar examples could be cited. Unfortunately, bogus stories and factoids abound on the net, posing an ongoing challenge for serious scholarship. One may hope that others will take heed: legitimate history has to be grounded in sources. Once these disappear, or get replaced by a vicious circle of web links, the stories may be entertaining, but they are no longer history. **D.E.R.**

Cardano's Favorite Problem: the *Proportio Reflexa*

Albrecht Heeffer

Cardano as a writer

Girolamo Cardano (1501–1576) is best known for his mathematics and in particular for publishing the formula for solving (some types of) cubic equations in his *Ars Magna* of 1545. Less well known is that Cardano was one of the most prolific "scientific" writers of the Renaissance. He composed almost two hundred works filling more than six thousand manuscript folios. These dealt with such diverse subjects as mathematics, astronomy, cosmology, geography, music, medicine, natural philosophy, astrology, prognostication, chiromancy (palm reading), metoscopy (reading faces), games and gambling, political administration, linguistics, Greek grammar, dialectics, natural and human history, the life of Christ, fables, hymns, the supernatural,... and he published an autobiography and a book on the books he wrote. On two occasions in his life he destroyed a number of his writings, but most of the unpublished manuscripts that were preserved were included in his *Opera Omnia*, edited by Gabriel Naudé in 1663.

Not only was Cardano an inordinately productive writer, he also had very specific ideas about the composition, structure and revision of texts. In his *Libris Propriis* he admits that he rewrote each of his texts at least three times. The *Libris Propriis* itself exists in six versions and was published in four different editions.[6] Some of his works, such as the *Ars magna* and *De subtilitate*, Cardano claims he revised more than fifteen times. He worked on his *On Subtlety* for a period of sixteen years and the book appeared in four published versions.[7] *On Subtlety* also contains some interesting thoughts on writing and structure of texts which are worthy of interest. Cardano believed there are ten styles of writing (*modi scribendi*) and 162 categories of knowledge (*argument*) an author can write about, so that all told there are 1620 different possible types of books. The organization of many of his works follows a predetermined structure and order. His motivation for writing *On Subtlety* and organizing it in 21 books came from one of his recurring dreams (*Libris Propriis*, [7] I, 108). Only after its first publication in Nürnberg did this dream cease to recur. In *On Subtlety* (book XVII) Cardano explains how he uses an indexing scheme to rearrange his texts (see Figure 1). This may very well be the first explicit reference in literature to an operative indexing data structure, as now used in computer science. Cardano first numbers the individual notes he used to compile a text. Then he assigns a 'utility' to each note depending on its merit. The result is shown in *tabula prima* with, for example, his first note getting the utility value of 7. Next, he sorts all his notes according to their utilities (as shown in *tabula secunda*), selecting the ones with highest utility for his new editions and leaving out the ones with the lowest scores.

In the course of half a century of writing books and publishing revisions, Cardano often returned to a previous subject to provide a new or an alternative explanation or story. In later works he even went so far as to disprove things he had accepted and even proved in earlier works. A

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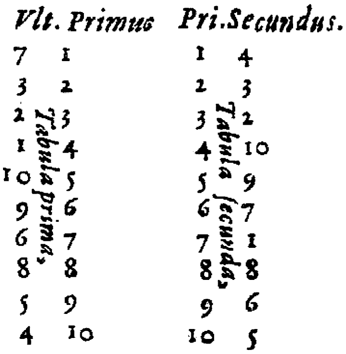
4 Girolamo Cardano, *Ars Magna or The Rules of Algebra*. T. Richard Witmer, trans., New York: Dover, 1968.

5 See footnote 8 on page 8 of his translation of *Ars Magna*.

6 For a modern edition of this book see [16].

7 The four versions are 1) Nürnberg 1550 with reprints in Paris in 1550 and 1551, Lyon 1551 and 1554 and Basel 1555 and 1557, 2) Basel 1554 with reprints in Lyon 1559 and 1580, this edition was translated into French and published in Paris in 1556, 1566, 1578, 1584 and Rouen in 1642, 3) Basel 1560 with reprints in 1581, 1582, 1611, 1664 and Lyon in 1663. 4) the edition in the *Opera Omnia* of 1663 with some minor differences. A long-awaited English translation of the full work by John M. Forrestor has just been published.



Vlt. Primus		Pri.Secundus.	
7	1	1	4
3	2	2	3
2	3	3	2
1	4	4	10
10	5	5	9
9	6	6	7
6	7	7	1
8	8	8	8
5	9	9	6
4	10	10	5

Figure 1. Cardano's indexing scheme for rearranging texts (from [8], III, p.626).

bizarre example is his 'refutation' of the rule of signs in *De Aliza Regulae* of 1570 (discussed in [9]). The most interesting example, his favourite problem, runs like a thread through his mathematical writings: the *proportio reflexa*.

Some background on construction problems

Before discussing what Cardano wrote on the *proportio reflexa* (or the reflexive ratio), it is useful to situate the problem in its proper historical context. Understanding Cardano's Renaissance state of mind is mandatory if one wishes to appreciate the importance he attached to his discovery.

The *proportio reflexa* is the ratio of the sides of a specific triangle inscribed in a regular heptagon, and thus useful for the construction of such polygons. It is called reflexive because, taking a larger and a smaller side (but not the largest and smallest), their ratio will equal the ratio formed by taking the sum of the smaller with the remaining third side to the larger side, as represented in equations (1) and (2) below. As such, the reflexive proportion is somewhat analogous to the golden ratio. We do not know how Cardano came to invent this notion, but as it is first mentioned in connection with the regular heptagon he must have found it while studying this classic construction problem.

Now, construction methods for regular polygons have a long history. The construction of a regular seven-sided polygon or heptagon is particularly interesting as it shares the property with the three famous problems of antiquity (quadrature of the circle, duplication of the cube, and the trisection of an angle) that they cannot be performed by ruler and compass alone. Traditional ruler-and-compass constructions assume the ruler has no markings and only one edge. The construction of a regular heptagon is possible, however, if one employs other means, in particular, by using a marked ruler along with a compass (a method known in antiquity as a *neusis* construction), or by using an ordinary ruler and an angle trisector. *Neusis* constructions were frequently employed in ancient Greek geometry, usually by means of a *neusis* ruler, a marked ruler which is rotatable around a given point.

While the *neusis* ruler and other alternative construction methods were well known among Renaissance geometers, traditional construction methods were often preferred. One geometer who strictly adhered to classical ruler-and-compass constructions was Johannes Kepler. The sacred geometry of regular polygons and solids, as propounded by Euclid, was deeply intertwined with Kepler's general philosophy of a harmonic universe. Indeed, he wrote a dedication to the first three Euclidean postulates for his magisterial work, *Harmonices mundi* from 1619. As pointed out by Henk Bos ([1], 183), Kepler's restrictive interpretation of exactness in geometry meant that he could only make use of such "harmonic ratios":

> The ratios of the sides of these polygons to the diameters of their circumscribed circles, were the crucial elements in Kepler's mathematics of harmony. They were harmonious ratios, and they could be known because these regular polygons could be constructed within a given circle by the Euclidean means of straight lines and circles. The regular heptagon, in contrast, was not knowable because it could not be constructed by straight lines and circles.

Kepler was well aware that the construction of the regular heptagon was beyond the scope of 'exact' methods. While he did not prove that the construction was impossible with ruler and compass, he did criticize the constructions known to him [13]:

> So no regular heptagon has ever been constructed by anyone knowingly and deliberately, and working as proposed; nor can it be constructed as proposed; but it can well be constructed fortuitously; yet it is, all the same [logically] necessary that it cannot be known whether the figure has been constructed or not.

One of the approaches to the regular heptagon criticized by Kepler was Cardano's *proportio reflexa*. In his *Harmonices mundi*, Kepler includes a study of the regular heptagon, in which he shows that he was familiar with Cardano's reflexive ratio, both from *On Subtlety* as well as the algebraic treatment in *De proportionibus*. Kepler's criticism of Cardano is based on three objections, two explicit and one implicit. He rejects Cardano's approach explicitly because the reflexive ratio does not comply with his criterion of exactness, as discussed above ([13], 62–3):

> This kind of proportionality seems to carry the implication that there is a unique precisely determinate proportion between the lines *EF* and *FB*; and Cardano, who when he discussed this matter concerning the side of the scalene trangle *BED*, gave it the name *proportio reflexa*, boasting falsely that he had found the side of the heptagon.

As argued by Bos [11], the scope of what Kepler considered to be legitimate geometry was vastly expanded by the end of the sixteenth century. This was especially due to the work of Viète and later Descartes, who used symbolic algebra as a tool for studying geometrical construction problems. Viète raised the *neusis* construction (and hence the trisection of an angle) to the level of a postulate. This led to a 'redefinition of geometrical exactness' which culminated in Descartes's *Geometry* of 1637.

A second objection raised by Kepler stemmed from his belief that the reflexive proportion was indeterminate and would lead to an infinity of solutions. In *Harmonices mundi* he reformulated the definition in terms of four quantities proportional two by two, for which he discusses some numerical examples ([13], 63–65). However, Cardano's algebraic approach leads to a determinate solution which can also be approximated numerically.

Kepler not only rejected unorthodox geometrical methods, he also displayed a disdain for the use of algebra ([11, 17] 189–193). He considered algebra as merely a set of practices useful for the merchant class, but far inferior to pure geometry. He thus implicitly rejected the reflexive proportion for methodological reasons, since he differed with Cardano, who placed geometrical and algebraic methods on the same footing. In some ways, Cardano was anticipating the transformation of geometry which began to take place toward the end of the sixteenth century.

As a last point, we should note that Cardano was less interested in the construction of the regular heptagon than he was in the special proportion he had discovered. Cardano's reflexive proportion shares the esthetics and mystery that surrounded other famous geometrical ratios such as π, the ratio of the circumference of a circle to its diameter, or φ, the golden ratio. While π and φ could be approximated with great accuracy by Renaissance mathematicians (Ludolph van Ceulen calculated π to 35 decimal figures), quantifying the reflexive proportion was a task that long eluded Cardano.

The *proportio reflexa*

The earliest reference to the *proportio reflexa* appears in an oration Cardano delivered at the Academia Platina in Milan in 1535; this was entitled *Encomium geometriae*. An *encomium* is a specific kind of oration in which praise is bestowed on some subject, in this case geometry. Cardano, however, adapted this rhetorical form as a way to discuss, the history and truth of mathematics, as he later did for other subjects [19]. Here he describes how the chords of a regular heptagon inscribed in a circle stand in the same relation as the paths of celestial bodies; these ratios can thus be constructed by geometrical means ([7], IV, 445). Book VI of the *Novae geometriae*, a lost work composed between 1534 and 1544, was fully dedicated to the *proportio reflexa*.[8]

The first edition of 1550 of his most popular work, *On Subtlety*, does not include the reflexive ratio. This edition does, however, contain a discussion of the peculiar ratio between the two-sided and three-sided diagonals of a regular heptagon.[9] One finds an exposition of the principle underlying the reflexive ratio in the second edition of 1554. This appears in book XVI, *On the Sciences*, after Cardano's description of geometry as "the most subtle of all sciences". Intended as an example illustrating that subtlety, he claims it as his own invention ("quae a nobis inventa est", [7], III, 598–600). In book 44 of his autobiography, entitled "Things of worth which I achieved in various studies", Cardano cites the reflexive ratio as his main achievement in geometry ([8], I, 39). But the most detailed explanation of the reflexive ratio comes from a work he wrote in 1568, *De proportionibus*, which was published in 1570 as part of *Regula Aliza*, concerning the construction of a regular heptagon. The 1554 and 1560 editions of *On Subtlety* already refer to the *Regula Aliza*.

In one of the few studies on Kepler's analysis of the heptagon, Judith Field sides with Kepler's criticism of Cardano's algebraic solution [14]. She claims that "Cardano somehow confused himself in his repeated use of 'reflective proportion'" and discerns "disconcerting flaws" in his mathematical reasoning. In the following we will demonstrate that this is an unfair representation by showing that Cardano's algebraic treatment of the reflexive proportion leads to a quartic equation that can be solved as an irreducible cubic. His reasoning is correct but also consistent with what one finds in his other writings.

In the *Encomium geometriae*, after observing that planets obey geometrical rules, he alludes to the principle underlying the related geometrical problem:[10]

> On the other hand, if three quantities of which the aggregate of the first and the third has a ratio to the second as the second has to the first, or as the ratio of the first and the second is to the third as the third is to the second.

In *On Subtlety* Cardano includes several drawings to illustrate the idea behind his geometrical demonstration (see Figure 2). The definition of reflexive ratio is illustrated by the triangle *ABC*. Simple reflexive ratio thus becomes

$$\frac{(AB+BC)}{AC}=\frac{AC}{BC} \qquad (1)$$

and equally

$$\frac{(AB+AC)}{BC}=\frac{BC}{AB} \qquad (2)$$

Cardano then constructs a second triangle *ABD* by bisecting $\angle ABC$ using line *BD*. Since the two triangles *ABC* and *DBC*

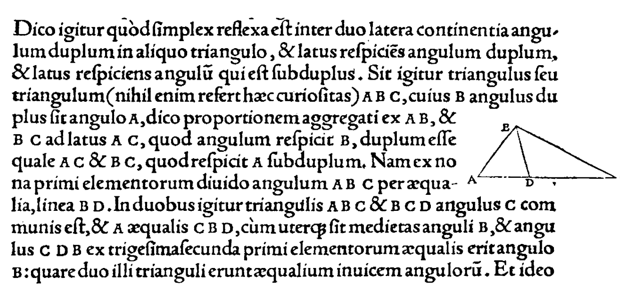

Dico igitur quòd ſimplex reflexa eſt inter duo latera continentia angulum duplum in aliquo triangulo, & latus reſpiciẽs angulum duplum, & latus reſpiciens angulũ qui eſt ſubduplus. Sit igitur triangulus ſeu triangulum (nihil enim refert hæc curioſitas) A B C, cuius B angulus duplus ſit angulo A, dico proportionem aggregati ex A B, & B C ad latus A C, quod angulum reſpicit B, duplum eſſe quale A C & B C, quod reſpicit A ſubduplum. Nam ex nona primi elementorum diuido angulum A B C per æqualia, linea B D. In duobus igitur triangulis A B C & B C D angulus C communis eſt, & A æqualis C B D, cùm uterq; ſit medietas anguli B, & angulus C D B ex trigeſimaſecunda primi elementorum æqualis erit angulo B: quare duo illi trianguli erunt æqualium inuicem angulorũ. Et ideo

Figure 2. Cardano's definition of the reflexive ratio (from *De subtilitate*, 1554, p. 427).

[8] Veronica Gavagna proposed the thesis that the extant *Commentaria in Euclidis Elementa* (manuscript Paris, BNF Lat. 7217) is an intermediate edition of this work [15].

[9] Veronica Gavagna brought this to my attention (personal communication).

[10] [7], 4, p. 445: "Rursus si trium quantitatum, quarum primæ & tertiæ aggregatum ad secundam ea ratio sit, quæ secundam ad primam: cum vero primæ & secundæ ratio ad tertiam, qualis terti ad secundam lineæ iungatur, circulusq; trigono circumscribatur, erit in hoc trigono tota heptagoni ratio absoluta:namq; prima, eademque minor linea, heptagoni latus est: secunda ac media, quæ duobus heptagoni lateribus subjicitur: tertia, quæ tribus ex una parte: quatuor autem ex alia heptagoni lateribus opponitur".

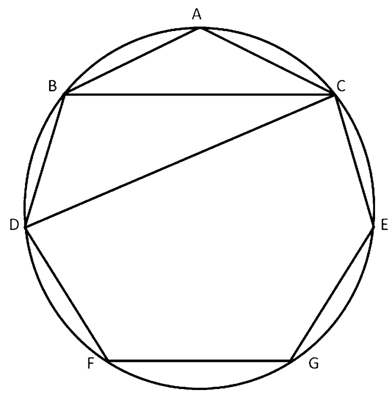


Figure 3. A triangle BCD in a regular heptagon (following [7], IV, 492).

have angle C in common, the proportion $AC : CB = CB : CD$, as can be demonstrated using the sixth book of *The Elements*.

Cardano next applies the principle to the sides of a regular heptagon inscribed in a circle (see Figure 3). It then follows that $(BD + CD) : BC = BC : BD$ and equally $(BC + BD) : CD = CD : BC$. Given the length of these sides, the question naturally arises as to how one might quantify this ratio for any given heptagon. In *On Subtlety* Cardano attempts an approximation to which we will return later ($AB = 9$, $AF = 16$ and $BF = 20$ in Figure 4). In a later work, *De proportionibus*, as well as in *Commentaria in Euclidis Elementa* and – following Gavagna – in the lost *Novae geometriae*, he tries to quantify the reflexive ratio by means of algebra. Let us first consider *De proportionibus*.[11]

Here he assumes that the side BD is equal to 1 and uses an unknown x (writing *pos.* in this case) for the side BC. Following definition 20 (which gives the reflexive ratio as $(BC + BD) : CD = CD : BC$) he arrives at $x + 1 : |CD| = |CD| : x$ which can be written as $|CD|^2 = x^2 + x$. Now, as $(BD + CD) : BC$ is in the same ratio as $BC : BD$ or $x : 1$, it follows that $1 + \sqrt{x^2 + x} = |BC|^2 = x^2$. Or else $x^2 - 1 = \sqrt{x^2 + x}$. Squaring both parts Cardano arrives at a biquadratic equation $x^4 - 2x^2 + 1 = x^2 + x$ to which he adds $4x^2$, resulting in:

$$x^4 + 2x^2 + 1 = 5x^2 + x \tag{3}$$

Cardano then claims that this reduces to the cubic equation

$$x^3 = \left(1 + \frac{3}{4}\right)x + \frac{7}{8} \tag{4}$$

This is the point where Judith Field, misquoting the equation as $(bc)^3 = (bc) + \frac{7}{8}$ discerns "disconcerting flaws". Cardano does not explain how he gets from (3) to (4), but this can be reconstructed from his treatment of biquadratics in the *Ars Magna*.[12] In chapter 39 of the *Ars Magna*, Cardano, duly acknowledging his student Lodovico Ferrari, lists twenty general cases of biquadratic equations and shows how these can be reduced to cubics. This is followed by a discussion of twelve problems, the ninth of which corresponds to (3). This also explains why Cardano uses (3) rather than the simplified equation $x^4 + 1 = 3x^2 + x$, as it is already in the required format since the left side $x^4 + 2x^2 + 1$ is the perfect square $(x^2 + 1)^2$. Cardano then adds $2y(x^2 + 1) + y^2$ to both sides. While on other occasions Cardano uses *co.* and *quan.* or *pos.* and *quan.* to distinguish two different unknowns (as we do by x and y) [8], he here only uses *pos.* adding to the confusion.

Adding the expression makes the left hand side of the equation a perfect square $(x^2 + 1 + y)^2$, the right hand side being $5x^2 + x + 2y(x^2 + 1) + y^2$ or $(5 + 2y)x^2 + x + (y^2 + 2y)$, which he also wants to make into a perfect square. This would be the case when the coefficients of the first term times the third term equals the square of half the second term. Therefore, $(5 + 2y)(y^2 + 2y)$ must be equal to $\frac{1}{4}$. In other words, we have to find a solution to

$$y^3 + \frac{9}{2}y^2 + 5y = \frac{1}{8}. \tag{5}$$

Cardano explains how to solve such equations (cube, square and first power equal to a number) in Chapter 17 of the *Ars Magna*. The procedure begins with a simple transformation which renders the coefficient of the squared term zero. In the present case, this is achieved by $y = z - \frac{3}{2}$, leading to the enigmatic 'reduction'

$$z^3 = \left(1 + \frac{3}{4}\right)z + \frac{7}{8}, \tag{6}$$

which is formally equivalent to (4).

In the *Commentaria in Euclidis Elementa* (and hence also in the lost *Novae geometriae*, which was composed earlier), Cardano uses a slightly different approach [15]. There he considers the side BD of the heptagon as the unknown, and the larger side BC as 1. This leads to $|CD|^2 = x^2 + x$ and $x^2 = 1 + |CD|$ and thus $x^2 = 1 + \sqrt{(x^2 + x)}$. By squaring both sides he arrives at the same biquadratic equation:

$$x^4 - 2x^2 + 1 = x^2 + x. \tag{7}$$

This time the reduction is done by dividing both sides by a common factor $(x + 1)(x^3 - x^2 - x + 1) = (x + 1)x$, so that Cardano arrives at the cubic equation[13]

$$x^3 - x^2 - x + 1 = 0. \tag{8}$$

He now uses the transformation $y = x - \frac{1}{3}$, as explained in Chapter 21 of the *Ars Magna* (cube and number equal square and first power) to arrive at the irreducible cubic:[14]

$$y^3 - \frac{7}{3}y + \frac{7}{27} = 0. \tag{9}$$

[11][7], p. 74: "Quare supposita db 1, bc 1 positione, erit dc latus [℞] 1 quad. p: 1 positione. (Per 20 diff.) Proportio verò, ut dictum est bd & dc ad bc, id est [1] p: [℞] 1 quad. p: 1 pos, ad 1 pos est, ut bc ad bd, id est 1 pos ad 1, igitur 1 p: [℞]v: 1 quad. p: 1 pos æquatur quadrato bc, quod est 1 quad. igitur 1 quad. m: 1 æquatur [℞]v: 1 quad. p: 1 pos quare 1 quad. quad. m: 2, quad. p: 1 æquatur 1 quad. p: 1 pos. Additis igitur communiter quatuor quadratis fient 1 quad. quad. p: 2 quad. p: 1 æqualia 5 quad. p: 1 pos. Et reducitur ad 1 cu. æqualem 1 3/4 pos p: 7/8". Two missing signs have been inserted between the brackets. Veronica Gavagna pointed out these typos in the *Opera* to me.

[12]Noted by Neyts [18] from an explanation by Hutton [12].

[13]The *Opus novum de proportionibus* ([7], p. 74–5) also contains another algebraic solution by Ferrari who does not start from the reflexive ratio but from Ptolemy's theorem and arrives at this same equation.

[14]This step is omitted in [15].

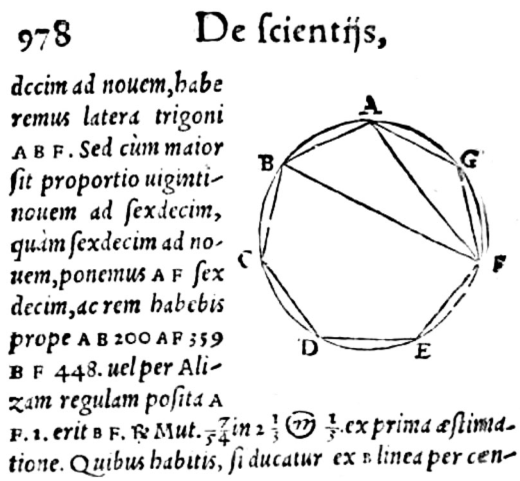


Figure 4. Approximating the reflexive ratio by triangle ABF in a regular heptagon (following *De subtilitate*, 1560, p. 978).

Quantifying the reflexive ratio

On two different occasions Cardano was able to express the reflexive ratio as the root of a cubic equation. However, as the roots of cubic and biquadratic equations can be complex, their exact quantification remained elusive. Within the historical context of the sixteenth century this must have seemed like a failure to Cardano. Abbaco algebra, practiced between 1300 and 1500, was obsessed by the search for exact quantitative solutions to problems. For example Maestro Antonio de' Mazzinghi, in a treatise composed around 1380, solves various commercial problems. For a certain bartering problem, he came up with a price of $\sqrt{43+\frac{6689}{10000}}+\frac{33}{100}$ *fiorini* [9]. Even more surpising is a problem discussed in Luca Pacioli's Perugia manuscript of 1478. The problem was to determine the number of men who will receive a certain amount of money ([10], problem PPM1260), and he arrives at $(9+\sqrt{101})$ for the number of persons. Within the whole corpus of these abbaco treatises on algebra one finds virutally no concern with approximate solutions.[15] The situation with regard to roots of cubic equations is quite different, however. In *On subtlety* Cardano attempts succesive approximations using the double reflexive ratio. He first takes $AB = 9$, $AF = 16$, and $BF = 20$ which leads to the approximation $1.8125 = \frac{29}{16} \approx \frac{16}{9} = 1.7950$. Increasing the values to $AB = 200$, $AF = 359$, and $BF = 448$, leads to the improved approximation:

$$1.8050 = \frac{648}{359} \approx \frac{359}{200} = 1.7950 \qquad (10)$$

$$1.2477 = \frac{559}{448} \approx \frac{448}{359} = 1.2479 \qquad (11)$$

The reduced biquadratic (3) and cubic (6) have the same roots. As the discriminant of the equation is less than zero, all the roots are real and have the approximate values

$$(1.801937736, -1.246979604, 0.4450418680)$$

of which the first corresponds with the ratio $BC{:}BD$ in the first analysis and its reciprocal is the unknown BD in the *Commentaria in Euclidis Elementa* (as shown in Figure 3).

Conceptual continuity

One may wonder – as I did – why Cardano considers the reflexive proportion as one of his greatest achievements, even though his main acclaim as a mathematician came from his *Ars magna*. He is especially famous in history because of 'Cardano's formula' for finding the roots of a cubic equation, but this was not his own discovery, as he duly acknowledges in the book. Ironically, Cardano's name is better represented in mechanics than in mathematics with the Cardan shaft, based on his description of this mechanical coupling in *On Subtlety*, as well as for other contrivances from this book such as the Cardano rings (also known as Chinese rings), and the Cardan grill, a cryptographic device.

Cardano was very conscious of his legacy as an author, which probably accounts for the frequent revisions he made of his writings. He was also preoccupied by the reception of his works and employed specific strategies for controlling his audience and as well as the criticisms his writings induced. In his autobiography *De vita propria liber*, chap. 48, Cardano compiled a list of no less than 73 authors who cited him favorably, such as Vesalius and Stifel. So his obsession with fame led to another invention: the first citation index in history. He also listed those who "contradicted him for the sake of making a reputation for themselves." This list included Scaliger, who wrote a critical response to *On Subtlety*, and Tartaglia with whom he fought the famous feud over the formula for solving the cubic. The editing, reiteration, reworking, and reassessing of previous work was typical for Cardano, and some ideas or problems, such as the *proportio reflexa*, run like a red thread through his work.

I sometimes find it useful to compare his approach to that of an artist rather than an author of scientific works. His frequent representations of a single idea in different contexts while using different approaches is typical for the work of an artist. The musician and composer Frank Zappa coined the term 'conceptual continuity' for his own music. Zappa's compositional approach was based on a compilation of reiterated and revised ideas rather than a consistent structure. Seen in this way, the *Proportio reflexa* provides a nice example of conceptual continuity in Cardano's thinking, and this may help explain why he was so fond of this particular finding. The reflexive proportion connects mathematics with the paths of celestial bodies while serving as a prime example of the subtlety of geometry. With the discovery of radical solutions to cubic and biquadratic equations it becomes expressible and solvable in terms of algebra. Finally, the difficulties in calculating the value of the proportion make it an elusive and mysterious quantity that seems to lie just beyond our grasp.

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[15] One rare exception is an interest problem discussed in [9].

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Cardano v. Tartaglia: The Great Feud Goes Supernatural

Tony Rothman

Words

Somewhere in Rome in October 1570, the Brescian mathematician Niccolò Tartaglia, "the stutterer," met with Aldo Cardano, son of Tartaglia's bitter enemy Girolamo Cardano. In return for promises of gaining an appointment as a public torturer and executioner, Aldo revealed to Tartaglia his father's whereabouts in Bologna. Tartaglia hastened to that city and had Cardano arrested on charges of heresy for having cast a horoscope of Jesus Christ.

If you have heard this story, or some version of it, you are far from alone, for it is to be found in well-known books and on prominent websites. If you believe it, you are in good company as well, because the same forums pass the story on without qualification. If you believe it, though, you have been hoodwinked, because it is complete and utter nonsense.

The story of the epic sixteenth-century feud between Girolamo Cardano and Niccolò Tartaglia over the solution to the cubic equation is justly one of the most famous in the history of mathematics. Its more colorful versions, involving the obligatory shifting alliances of the sixteenth century, subterfuge, betrayals, and secret dossiers—let's not forget poison and syphilis—fairly scream for a theatrical presentation. Even a slightly sober investigation, however, shows a less than Borgian scenario. The "cubic affair" in fact becomes a prime example of how scientific folktales, which have little or no basis in the historical record, nevertheless get passed up the great chain of existence until they become enthroned in the eighth heaven of print or cyberspace. In the case of the Great Feud, we are privileged to be able to trace the progress of the tale in an apparently straightforward manner. Nowhere in the strictly scholarly works on Cardano, for instance those by James Eckman,[16] Anthony Grafton,[17] or Nancy Siraisi,[18] will one find any of the aforementioned lurid details, or indeed in

[16]James Eckman, *Jerome Cardan* (Johns Hopkins: Baltimore, 1946).

[17]Anthony Grafton, *Cardano's Cosmos* (Harvard: Cambridge, 1999).

[18]Nancy Siraisi, *The Clock and the Mirror* (Princeton: Princeton, 1997).

the standard nineteenth-century account, Henry Morley's loquacious two-tome biography of the astrologer-physician.[19] In the Italian literature on Tartaglia, the biographies of Masotti[20] and Gabrieli[21] for example, any vengeful machinations—murderous or merely injurious—are equally absent. Even Oystein Ore's semipopular work on Cardano,[22] which though lacking references and unblushingly biased in Cardano's favor, more or less adheres to known facts and avoids descent into sensationalism. A discontinuity occurs when one passes to the ultraviolet end of the spectrum. There, aboard more popular retellings, Hal Hellman's *Great Feuds in Mathematics*,[23] and Alan Wykes's *Doctor Cardano, Physician Extraordinary*,[24] one decisively abandons the world of documents and evidence for realms unknown.

That tabloid histories have supplanted mundane reality in numerous essays suffixed by *.edu* is perhaps less surprising than it is sad or amusing, depending on your momentary disposition. Scientists, we must face facts, are suckers. Beyond the hermetic world of scientific discourse, a significant percentage of folks fail to observe our clerical vows to facts, data, natural law, and logic, and the same folks aren't above pulling a fast one. It is not for nothing that magician James Randi in his investigations of pseudoscientific claims has steadfastly advocated that one needs someone trained in uncovering deception, not someone whose second nature presumes honesty. Scientists, trusting souls, can be ruled out. In their naivety they also, perhaps even more than most children, love a good story. When a tale comes around that satisfies our analytical lust for the three C's: completeness, consistency, and contingency ("no-plot-holes storytelling"), scientists' inherent gullibility leads us to accept it without question, in particular when it's too good to be true.

Absolute Truth (More or Less)

The tales surrounding the Great Feud do cry out loudly for a theatrical release; indeed my initial impetus to investigate them was to write a play about the episode, which—despite the stubborn intrusion of reality—I subsequently did, titling it *The Great Art*. Much of the first half of what people believe they know about the famous affair is in fact true.[25] The outstanding mathematical challenge of the early 1500s was to solve the cubic equation, in other words to find a "cubic formula" analogous to the famous quadratic formula, which had been known since antiquity. By contrast, the cubic formula had eluded all attempts to find it and most mathematicians of the era, following Fra Luca Pacioli, believed that such a solution was beyond the powers of human reason.

The Italian university system at the time curiously resembled our own, with tenure nonexistent and itinerant professors eking out an existence on temporary appointments. In such a milieu an important means of advancement were public "challenge matches," mathematical, medical, and otherwise, which incidentally proved extremely popular with the citizenry. In 1535, mathematician Antonio Maria Fiore challenged Niccolò Tartaglia (1499–1557) to such a contest. Mysteriously, Fiore had been boasting that he was in possession of the solution to the "depressed cubic," that is an equation of the form $x^3 + ax = b$, where a and b are positive numbers. (At the time, the concept of a solution to the general cubic $ax^3 + bx^2 + cx + d = 0$, for any real coefficients, had yet to arise. Numbers reflected the positive physical world and hence negative numbers were highly suspect. The equation $ax^3 + cx + d = 0$ was thus regarded as completely different from $ax^3 + cx = d$, which in turn was completely different from $ax^3 + bx^2 = d$. There were thirteen cases in all, which needed to be solved separately.)

Fiore posed to Tartaglia thirty problems, all of which boiled down to the depressed cubic. ("A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is the profit?"[26]) Several years earlier, Tartaglia had discovered how to solve the case $ax^3 + bx^2 = d$ and on the night of February 12–13, 1535, he perceived the solution to the depressed cubic as well. Tartaglia was thus able to solve all of Fiore's problems within 2 hours and, for his own part, having posed problems that Fiore could not solve, easily won the match. Tartaglia declined the thirty banquets that were the stakes of the contest.[27]

News of Tartaglia's victory spread throughout Italy and in 1539 Girolamo Cardano (1501–1576), who was preparing a book on mathematics, approached Tartaglia with a request for his solution. After strenuous refusals Tartaglia finally relented when the two met in Cardano's house in Milan, on condition that Cardano never publish it. Cardano swore a sacred oath that he would not. However, in 1543 he and his student Ludovico Ferrari (1522–1565) learned that Antonio Fiore had obtained the solution from his own teacher, Scipione del Ferro (1465–1526), who had discovered it three decades earlier, but never published it. Feeling released from his vow, Cardano published the solution, with considerably more praise for del Ferro than Tartaglia, as well as solutions to the other cases, in his 1545 book the *Ars Magna*,[28] which became the most important mathematical treatise of the sixteenth century.

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[19] Henry Morley, *Jerome Cardan* (Chapman and Hall: London, 1854).

[20] A. Masotti, *Niccolò Tartaglia*, in *Storia di Brescia*, II, pp. 587–617 (1963).

[21] Giovanni Battista Gabrieli, *Nicolò Tartaglia, Invenzioni, Disfide e Sfortune* (Brescia: 1986). This is the most complete account of Tartaglia's life I have found.

[22] Oystein Ore, *Cardano the Gambling Scholar* (Dover: New York, 1965).

[23] Hal Hellman, *Great Feuds in Mathematics: Ten of the liveliest disputes ever* (John Wiley: Hoboken, 2006).

[24] Alan Wykes, *Doctor Cardano, Physician Extraordinary* (Frederick Muller: London, 1969).

[25] See Ore's chap. 3 or the St. Andrew University MacTutor History of Mathematics website (henceforth MacTutor): http://www-history.mcs.st-and.ac.uk/HistTopics/Tartaglia_v_Cardan.html.

[26] *The History of Mathematics: A Reader*, John Fauvel and Jeremy Gray, eds. (The Open University: London, 1987), p. 254.

[27] Gabrieli provides numerous excerpts from the Tartaglia-Cardano dispute. For English-language excerpts see Fauvel and Grey, pp. 254–256 ; MacTutor and Ore.

[28] Girolamo Cardano, *Ars Magna, or The Rules of Algebra*, translated by T. Richard Witmer (Dover: New York, 1993).

At that point, one might say without exaggeration that all hell broke loose. Tartaglia, in his own book *Quesiti et Invenzioni Diverse* (*Various Questions and Inventions*) of 1546, accused Cardano of theft and a violation of a sacred trust (or perhaps a financial one—challenge matches were after all worth good money). Cardano, by then Italy's most famous physician and astrologer, evidently did not want to enter into a public dispute with Tartaglia and turned the matter over to Ferrari, who very publicly challenged Tartaglia to a contest. Vicious manifestos flew back and forth between the two for 18 months. "You make up proofs in your own head and thus they usually have no conclusion." "I truly do not know of any greater infamy than to break an oath, and this holds not only in our own, but in any other religion." "With these lies you attempt to convince the ignorant that your statements are true." "I honestly expect to soak the heads of both of you in one fell swoop, something that no barber in all Italy can do." "You are a devil of a man, wanting to be an inventor when you have the head of an adder, which can understand nothing."[29]

Apart from reputation, under dispute were thirty-one questions each combatant had proposed to the other on algebra, geometry, and philosophy. On August 10, 1548, the two antagonists and a large crowd of Ferrari's supporters met at The Church in the Garden of the Frati Zoccolanti in Milan for the final judging, presided over by the governor of Milan himself. No record exists of what exactly transpired during the occasion. It is generally accepted that Ferrari won, because Tartaglia slipped away during the first night, although from Niccolò's recollections one might conclude that he couldn't get a word in edgewise.

Disputable

All of this is fairly well documented: the *cartelli* and problems exchanged between Tartaglia and Ferrari exist, and in his books Tartaglia gives verbatim accounts of his letters and meeting with Cardano.[30] There is little reason to suspect that Tartaglia's version is far from the truth: Apart from the fact that Niccolò appears to have been a pack rat until the end of his life, Cardano in effect never disputed his claim in the *Ars Magna*. Some authors, for example Witmer[31] and Hellman,[32] argue that Ferrari (who was present at the meeting between Cardano and Tartaglia) later vociferously denied that Cardano had ever sworn such an oath. I find no evidence that this is the case. The relevant passage is from Ferrari's second *cartello*:

> First of all let me remind you, so that you don't remain astonished and wonder where I have heard all your lies, as if by a revelation of Apollo, that I was present in the house when Cardano offered you hospitality and I attended your conversations, which delighted me greatly. It was then that Cardano obtained from you this bit of a discovery of yours about the cube and the *cosa* equal to a number,* and this languishing little plant he recalled to life from near death by transplanting it in his book, explaining it clearly and learnedly, producing for it the greatest, the most fertile and most suitable place for growth. And he proclaimed you the inventor and recalled that it was you who communicated it when requested.
>
> What more do you want? "I don't want it divulged," you say. And why? "So that no one else shall profit from my invention." And therein, although it is a matter of small importance, almost of no utility, you show yourself un-Christian and malicious, almost worthy of being banned from human society. Really, since we are born not for ourselves only but for the benefit of our native land and the whole human race, and when you possess within yourself something good, why don't you want to let others share it? You say: "I intended to publish it, but in my own book." And who forbids it? Perhaps it is because you have not solved it entirely....[33]

Polemics one sees in abundance; an oath or its denial, no. Robert Kaster of Princeton University has graciously checked the facsimile of the entire Latin original for me and finds no mention of the oath elsewhere. Nor does Ore, who presents this translation, claim any denial of oath on Ferrari's part. To all appearances it is merely Ferrari's justification, on the part of the human race, for Cardano's publication of the cubic formula, and his admonition to Tartaglia to stop kvetching.

Hellman also gives credence to Alan Wykes's claim in his book *Doctor Cardano, Physician Extraordinary* that Cardano in fact worked out the formula for himself and then by "a slip of pen or memory, he wrote that Tartaglia had communicated the discovery to him and given him permission to use it."[34] Wykes, in a manner that will become familiar, provides no justification for this fabulous assertion, which requires that Tartaglia invented not only the meeting between himself and Cardano, but also their

[29] Facsimiles of the original manifestos, first published by Enrico Giordani in 1876, are now available online in Latin and old Italian at http://books.google.com/books?id=rBB1mTyvRDsC&source=gbs. For excerpts in Italian, see Gabrieli or Luigi di Pasquale, "I cartelli di matematica disfida di Ludovico Ferrari e i controcartelli di Nicolò Tartaglia," I, *Period., Mat*. (4) 35, 253–278 (1957); II, *Period., Mat*. (4) 36, 175–198 (1957). For English excerpts see Ore.

[30] See sources already cited.

[31] See Witmer's preface to the *Ars Magna* [2], p. xviii, note 26.

[32] Hellman, p. 18.

*In sixteenth century Italy, the unknown was referred to as the *cosa* (the thing). "The *cosa* and the cube equal to a number" was therefore the expression for the depressed cubic $x^3 + ax = b$.

[33] With some minor corrections from Robert Kaster, this is the translation given by Ore, p. 94.

[34] Wykes, p. 115.

entire correspondence. It also makes Ferrari's previously mentioned eyewitness account impossible. For the record, in the *Ars Magna* Cardano writes, "[Tartaglia] gave [the rule] to me in response to my entreaties, though withholding the demonstration."[35]

Oath aside, to this day the larger discussion centers on whether Cardano's actions were justified, given that Tartaglia had failed to publish his results in the decade after his contest with Fiore. I intend to avoid that particular debate. For diverse opinions the reader may want to see Eckman's detailed study.[36] (On the matter of the oath, Eckman writes, "There is, of course, no doubt as to the breach of faith on the part of Cardan. It was flagrant, even if allowance is made for the moralities of the sixteenth century in respect to mutual relationships."[37]) I do point out that statements, beginning with Ferrari's, to the effect that Tartaglia stood against the progress of science by intending to keep his discovery secret, appear grounded less in reality than in rhetoric. Even in the midst of his diatribe Ludovico recollects that Niccolò had protested only that he wanted to publish it himself. To be sure, in 1539 Tartaglia had said to the bookseller Zuan Antonio de Bassano, who acted as intermediary between himself and Cardano, "Tell *Eccellenza* that he must pardon me: when I propose to publish my invention, I will publish it in a work of my own, not in the work of another man, so that *Eccellenza* must hold me excused."[38]

Of course Tartaglia did not publish; nevertheless his excuse was evidently plausible: for many years he was occupied with the first translation of Euclid into any living language (Italian, 1543), and a modern edition of Archimedes (1544), both signal events in the history of mathematics. Indeed, in 1541 he wrote to his English pupil Richard Wentworth, assuring him that he would publish his formula after these works were completed.[39] Tartaglia may have also lost his entire family at about the same time.[40] And then Cardano beat him to the punch.

As it turns out, a year after the appearance of the *Ars Magna*, Tartaglia published his *Quesiti*, where one finds this striking passage in the dedication:

> I reflected that no small blame is attached to that man who, either through science, his own industry or through luck, discovers some noteworthy thing but wants to be its sole possessor; for, if all our ancients had done the same, we should be little different from the irrational animals now. In order not to incur that censure, I have decided to publish these questions and inventions of mine.[41]

Unless one believes that this statement was forced by publication of the *Ars Magna*, it does not appear to be of a man unwilling to divulge his results.

Regarding the allied view, implicit in Ore's work, that Tartaglia's position made him the last "medieval man" who put personal gain over communal progress, one might at this juncture bemoan the fact that those writing about the feud have been mathematicians rather than physicists. Tartaglia's first book, the *Nova Scientia* of 1537, was in fact the earliest attempt to treat the trajectory of projectiles by mathematical means, and it surely provided the model for Galileo's later *Two New Sciences*. In the *Quesiti*, Tartaglia became probably the first natural philosopher to openly challenge Aristotelian mechanics. It is interesting that while Cardano's publication of the cubic formula resonates with today's "open source" culture, Tartaglia's reasons for hesitating to publish his results on ballistics ("it was a blameworthy thing…a damnable exercise, destroyer of the human species…[and] I burned all my calculations…[42]) might have been written by today's antinuclear movement. Only under threat of a Turkish invasion did Tartaglia change his mind. It is also curious that the St. Andrew University MacTutor History of Mathematics website, which is fairly comprehensive, does not even mention Tartaglia's major work, the *Trattato Generale di Numeri et Misure* of 1556, usually considered one of the most important textbooks on arithmetic of the sixteenth century.

Falsifiable

If scholarly DNA requires arguments about everything, one thing is fairly impervious to even academic genetic coding: After the face-off between Tartaglia and Ferrari in 1548, the historical record rapidly grows mute. Little is factually known about Tartaglia's life apart from the occasional public document and autobiographical passages scattered throughout his mathematical works. As we know, however, Nature abhors a vacuum, and it may well be the vacuum that has inspired authors to fill it with tales that extend the feud to literally the supernatural domain.

It is true, as Ore relates, that after the misadventure with Ferrari, the patrons who in early 1548 had invited Tartaglia to his native Brescia to lecture on Euclid did an about-face and refused to pay him for his labors. Niccolò lost 18 months' salary and was forced to return to Venice, where he had lived since 1534, and continue his livelihood as a private mathematics teacher. But as plausible as it might seem that his hosts' bad faith was the result of his poor showing in Milan[43]—contingency, after all—there is no documentary evidence that this is the case. In fact,

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[35]Cardano, *Ars Magna*, p. 96.

[36]Eckman, chap. 4.

[37]Ibid., p. 64.

[38]Ore, p. 66, and MacTutor.

[39]*Mechanics in Sixteenth Century Italy*, translated and annotated by Stillman Drake and I. E. Drabkin (University of Wisconsin: Madison, 1969); *Metallurgy, Ballistics and Epistemic Instruments*, The Nova Scientia *of Nicolò Tartaglia, a new edition*, Matteo Valleriani et al., eds. (Edition Open Access: Berlin, 2013): http://www.edition-open-access.de/sources/6/index.html.

[40]Drake and Drabkin, p. 21; Gabrieli, p. 20.

[41]Drake and Drabkin, p. 99.

[42]Ibid., p. 68.

[43]Ore, p. 105.

Tartaglia continued to lecture in Brescia for another year after the historic showdown. For this reason Gabrieli argues that the two events are unconnected.[44]

There can't be any doubt that Tartaglia remained extremely bitter about what had transpired, and even in his last work, the *Trattato Generale*, he returned to the problems posed a decade earlier in the manifestos, making scornful remarks about his opponents' solutions. Nevertheless, all stories—*all*—that Tartaglia devoted the remainder of his life to revenging himself against his nemesis are apocryphal, in the original sense of the word, or plainly false. The most recent retelling is Hal Hellman's 2006 *Great Feuds in Mathematics*,[45] already mentioned, which I now quote at length because it provides a concise compendium of what have become the standard rumors and legends surrounding the Cardano-Tartaglia affair. By the mid-sixteenth century, the Roman Inquisition and Counter Reformation were underway. In the decades after the Ferrari-Tartaglia contest,

> …Scholars of all sorts were under suspicion, but somehow Tartaglia had managed to place himself satisfactorily. Cardano could find no employment and, according to Wykes, "it was Tartaglia who was the instigator of most of the refusals that met him in College and University. It was simple enough, with the network of the Inquisition flourishing in city, vineyard, village and public square, to keep a shadowy hand on the shoulder of any citizen, great or small."
>
> This was just the warm up, though. On October 13, 1570, almost a quarter of a century after publication of *Ars Magna*, Tartaglia served up a double blow. Using Cardano's own son Aldo as in informant as to Cardano's whereabouts, Tartaglia handed him to the Inquisition. Tartaglia had been collecting evidence against Cardano for years. Among this "evidence" was Cardano's rejection of the pope's invitation that he become the pope's astrologer and physician. Tartaglia pointed to the "sarcasm" evident in Cardano's comment that "His Holiness by his study of astrology has surely raised himself among the greatest of such scientists and has no need of help from such as myself."
>
> Cardano's horoscope of the life of Jesus was also damning, as were a variety of other statements that, taken out of contexts, could be construed as blasphemous. In one of his publications, for example, he had suggested that God is a universal spirit whose benevolence is not restricted to holders of the Christian faith. Today he might be admired for such an ecumenical statement; at the time it was apparently a dangerous idea.
>
> And so it went. Cardano, fortunately, was not subjected to torture or put to death, but he was thrown into jail. He sought desperately for help and was able to reach out to an official in the church, Archbishop Hamilton, who had in the past asked to be called upon if need be. The archbishop came through for Cardano, who was released a few months later. It was just in time, for not long after, the archbishop's own fortunes changed; he was captured by the forces of Mary, Queen of Scots, and beheaded.
>
> Tartaglia finally had had his revenge. Cardano lived on in obscurity in Rome, where he worked on his autobiography, which is one of the works that has come down to us in full. He probably never knew, and just as well, that his daughter Chiara had died of syphilis, and that it was Aldo who betrayed him to the Inquisition and who was rewarded with an appointment as official torturer and executioner in Bologna.
>
> Cardano died on September 20, 1576. Less than a year later, Tartaglia followed him to the grave.

As signaled in the introduction, the same stories, that "Cardano himself was accused of heresy in 1570 because he had computed and published the horoscope of Jesus in 1554," and that "apparently, his own son [Aldo] contributed to the prosecution, bribed by Tartaglia," can be found in the Wikipedia entry on Cardano.[46] The contention that his daughter Chiara died of syphilis is so widespread on the Internet that specific references are unnecessary. According to one essay, the tragedy prompted Cardano to write one of the earliest treatises on the disease.

What truth to these tales? First, the contention that Cardano was unable to find a job, whereas Tartaglia "managed to place himself satisfactorily," is completely counterfactual. The 1550s saw Cardano at the height of his fame, with a professorship in Pavia, at least one genuine bestseller (*De Subtilitate*) and invitations by European potentates (e.g., Archbishop Hamilton of Scotland, whom Cardano cured of asthma to great acclaim). Throughout the 1560s, Cardano remained relatively prosperous, although he resigned from the University of Pavia, evidently because of accusations of pedophilia,[47] and moved to a lectureship at the University of Bologna. Tartaglia, on the other hand, returned to Venice in poverty and remained desperately poor until his death, bequeathing only books to his publisher, brother, and sister, as well as a few household items to the latter.[48] (Niccolò had hardly been interred before the publisher, alas, made off with all the books.) Cardano was indeed arrested in Bologna on October 13, 1570, for impiety, although nowhere in his writings does he disclose the reasons, and no records of the proceedings have come to light. It is possible that his arrest resulted from his 1554 horoscope of Jesus Christ, but this has never been

[44] Gabrieli, p. 85.

[45] Hellman, pp. 23–24.

[46] http://en.wikipedia.org/wiki/Gerolamo_Cardano. [Note added: Since posting the preprint to this article, the erroneous information in the Wikipedia entry has been removed.]

[47] Grafton, p. 188. The matter is also discussed obliquely in Cardano's *Book of My Life*, pp. 96–99; see note 55 that follows.

[48] Gabrieli, pp. 104–110.

established, even if it is consistent.[49] He was released from prison after 3 months because of intervention by his friends Cardinals Morone and Borromeo, held under house arrest for a time, then invited to Rome, where he spent the last 5 years of his life provided for by the pope, continuing to practice medicine but no longer allowed to teach or publish.

As for the remaining stories, tales of Aldo Cardano's complicity in his father's arrest have been alternately surfacing and submerging since at least the nineteenth century,[50] but Tartaglia's role in the affair has been most notably propagated, if not altogether invented, by Alan Wykes, whose *Doctor Cardano, Physician Extraordinary* Hellman follows. In Wykes's account not only can Tartaglia, a poor Venetian mathematician ("in whom the seeds of instability had been nourished by childhood environment and had grown into weeds choking the flowers of his own brilliance") worm his way into the good graces of the governor of Milan in order to thwart Cardano's advancement, but in doing so he is able to subvert his enemy by disclosing the horoscope of Jesus to a papal emissary.[51] Wykes's plotting is impressive and lurid. Too impressive, too lurid. I have reluctantly come to the conclusion that his work was either written from memory without double-checking sources, or is a deliberate literary hoax. One should of course think twice before imputing motive, but were Wykes alive, I would certainly ask him to explain himself.

It is easiest to deal with Wykes's book by beginning at the end. The closing sentences are: "[Cardano] died on 20th September 1576, a man not without greatness in an age of great and cruel men. Less than a year later his enemy Tartaglia died also."[52]

In fact, Tartaglia died on the night of 13–14 December, 1557, 19 years before Cardano. This is not a matter of conjecture or debate: his Last Will and Testament exists and has been published; Tartaglia was buried in the church of San Silvestro in Venice according to his wishes.[53] Of course, Wykes's error makes most of the previous claims impossible, by chronological protection. It is nevertheless instructive to see how he justifies, for example, the tale that in 1570 Tartaglia bribed Cardano's son Aldo into turning his father over to the Inquisition. Wykes writes:

> The boy Aldo, to whom I had promised the reward of the appointment of public torturer and executioner in that city [Bologna], came to me in Rome with the intelligence that his father was in Bologna, awaiting an interview with the syndics. I thought to myself, 'Ah! This will be pleasant, to raise his hopes that at last the restrictions are about to be lifted from him and then, an instant before the realization of those hopes to cast him into prison. And so it was. I hastened to Bologna, and there he is still sheltered, in the ruins of a hovel, awaiting an ascent to his former status. I instructed the guards to arrest him as he set out for his appointment.[54]

The context makes clear that Wykes intends the reader to believe Tartaglia wrote this passage. The fact that in 1570 Tartaglia had been dead for 13 years should be sufficient reason to doubt it. Additionally, there is no evidence that Niccolò was ever in Rome, Cardano's presence in Bologna was hardly a secret, and his presence in a hovel, really?—he had been awarded the high honor of being made a citizen of the city. Who then wrote the passage? In Wykes's book, it is tagged "footnote 2" for Chapter 18, but in the endnotes for that chapter, a source for footnote 1 is listed and nothing more. Reference 2 is simply missing. Given that I have found it virtually impossible to confirm a single citation in Wykes's book, I would not be surprised if he invented it himself.*

Here I must turn to Cardano's autobiography, *De Vita Propria Liber*, or *The Book of My Life*,[55] which is one of the Renaissance's most famous memoirs and the work through which we know most about the author. In it Cardano is remarkably frank about his failures as a father and the disasters of his two sons, the elder Giambattista (1534–1560) who was executed for poisoning an adulterous wife, the younger Aldo (1543–?), who was arrested on numerous occasions for theft. After Aldo burglarized his father's own home in 1569, Cardano had him imprisoned and disinherited him. Wykes makes extended assertions[56] that Aldo acted as a torturer and executioner and that Cardano knew it via public accounts ("Messer Aldo Cardano, executioner, for torturing by rack and vice, Valentino Zuccaro, 3 scudi."), but nowhere in *The Book of My Life* does Cardano mention any such activities. The only source Wykes offers for his claims is Cardano's *De Consolatione*, which was published when Aldo was negative 1 year old.[57] A precocious child indeed.

A similar haze surrounds Cardano's daughter, Chiara (1536–?). Wykes writes that by the age of 16 Chiara had seduced her elder brother Giambattista.[58] No reference is given. He does present a single-sentence quotation "There was nought of honesty at all in her whoring," which points us to Peter Martyr Vermigli's *Loci Communes*.[59] He next gives an

[49]See Eckman, p. 33 *et seq*.
[50]Ibid., pp. 32–33.
[51]Wykes, p. 117, pp. 120–121.
[52]Ibid., p. 176.
[53]Gabrieli, pp. 104–110.
[54]Wykes, p. 174.
*The inadequate citations throughout Wykes's book make it extremely difficult to verify anything. The few citations that are given are to titles only and never include page numbers. Ore also fails to provide references, and certain quotations appear to me dubious (e.g., the unending adjectival string by which Cardano describes his own character on Ore's p. 25 is not to be found in Cardano's *Book of My Life* (next footnote)). When I have been able to track down others, however, they appear reasonably accurate.
[55]Girolamo Cardano, *The Book of My Life*, translated by Jean Stoner (New York Review Books: New York, 2002). This translation originally appeared in 1929.
[56]Wykes, pp. 151–152.
[57]http://books.google.com/books/about/De_Consolatione.html?id=evs5AAAAcAAJ.
[58]Wykes, p. 142.
[59]Ibid.

extended excerpt from a letter by Chiara's husband Bartolomeo Sacco, in which Sacco writes, "Not only have you shed upon me the great pox in the person of your unclean daughter, but you have given me a wife whose demands night and day are more than can be met by the staunchest lover of couch pleasures…"[60] The missive becomes far more graphic, ending with the husband's threat to seek an annulment of the marriage. Again, no reference is given. Wykes does provide a source for two subsequent passages regarding Chiara: "A young woman still, she was brought to book of the Spanish disease and her own sad flux." Chiara's sterility was due to her incestuous relationship with Giambattista and "the exaction of the price" by the ecclesiastical courts for this crime "was endless." The citation for the quoted passages is Cardano's *Book of My Life*.

What are we to make of all this? One might scratch one's head for a moment to ask why Peter Vermigli, a famous Florentine theologian, would be writing about Chiara Cardano, yet alone in a compendium of theological practices. In answering this question I am limited by my inability to read Latin, but I have checked all the English translations of Vermigli's works at Princeton University (which do not include the *Loci Communes*) and there are only two passing references to Girolamo Cardano and none to Chiara. The *Loci Communes* itself is now available online as a Google Book.[61] Its index contains no mention of Cardano or his daughter.*

As for Wykes' references to *The Book of My Life*, we are immediately confronted by Cardano's own statement, "From my daughter alone have I suffered no vexations beyond the getting together of her dowry, but this obligation to her I discharged, as was right, with pleasure."[62] In fact, I challenge anyone to find the passages Wykes cites in *The Book of My Life*. Initially, I assumed that he must have worked from a more complete edition, but in his bibliography Wykes lists the translation he used as the one by Jean Stover [sic]. The 1929 translation by Jean Stoner is the only one into English of which I am aware. Under normal circumstances I would assume this was a simple misprint; in light of the rest…Neither does Morley in his biography of Cardano mention any such behavior on Chiara's part.

In a word, I have found only one "documented" contention apart from Wykes's that Chiara Cardano ended her life as a prostitute or died of syphilis. Eckman does cite H. Kümmel as writing in 1910, "*Eine Tochter, das einzige Kind, das ihm geblieben war, brannte mit einem Galan durch und endete als Dirne*,"[63] or, "A daughter, the only child left to him, eloped with a gallant and ended up being a prostitute." On the other hand, Eckman himself says of this passage that he knows of no authority for it. Given that Chiara married Bartolomeo Sacco, a patrician, almost certainly before the troubles with Girolamo's sons began, it is difficult to see what authority there could be.

To summarize, as far as I am able to determine, all the direct quotations in Wykes's book from family and household members are either loose paraphrases from Cardano's *Book of My Life* or fabrications. And incidentally, Cardano neither invented, nor claimed to invent the universal joint, or Cardan shaft—another popular pass-me-down that can be found in Wykes's book[64] and on Wikipedia—but only a chair that could be kept level on an incline.[65]

More Words

At this point I trust that I have presented enough evidence to throw serious doubt on most of the standard stories surrounding the Cardano-Tartaglia affair. The exercise has not been, however, merely to bring to light careless errors in the popular and semipopular literature. Mistakes, after all, are inevitable. If, however, we extend the concept of scholar to include writers and editors, to any profession that strives toward getting at truth rather than hoodwinking an audience, then it seems to me that such callings require not only intellectual honesty, but intellectual discipline and a basic attention to detail, where the devil resides. The fact that, on the one hand, the Wikipedia editors get the date of Tartaglia's death correct, but on the other hand repeat the story that he abetted Cardano's arrest, tempts one to laugh. As mentioned in the introduction, the apocrypha I've discussed never seem to be repeated in the more scholarly works about Cardano that concern his astrological or medical activities. The tales are apparently confined to the mathematical sphere. Alan Wykes may not have been a mathematician, but many of his readers seem to be.

Experience forewarns that a nonnegligible percentage of readers will meet the present essay with a shrug and reply that legends, at least great ones, are preferable to mundane "true" stories. A first answer is that, yes, great legends confer moral truths. In this case of the Great Feud I do not see any deep truths, only negligence, deception, and mean-spiritedness. I can provide a second answer by recounting yet another tale: Thirty years ago I published an investigation on the various myths surrounding Evariste Galois.[66] Intending to announce my findings at a seminar at the University of

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[60]Ibid., p. 149.

[61]Peter Martyr Vermigli, *Loci Communes*, http://books.google.com/books?id=Hgl-AAAAcAAJ&printsec=frontcover&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false.;

*Given that *De Consolatione* and *Loci Communes* mean, respectively, "On Consolation" and "Commonplaces," one wonders if there is some hidden joke here on Wykes's part.

[62]Cardano, *Book of My Life*, p. 82.

[63]Eckman, p. 33.

[64]Wykes, p. 108.

[65]Eckman, p. 77.

[66]Original version: Tony Rothman, "Genius and Biographers: The Fictionalization of Evariste Galois," Amer. Math. Mon. 89, 84 (1982). Revised version available online at various locations.

Texas, I thought it would be appropriate to wear a period costume for the occasion and betook myself to the drama department. The wardrobe mistress didn't have anything available from the proper timeframe, and so I asked her just to give me a nice ruffled shirt. At this she took offense, saying that I was concerned only with historical accuracy in science, not in costumes. She did relent and lent me a beautiful shirt, but the lesson was a good one and has remained. Scientists only reluctantly acknowledge truth in other fields, but standards are standards. If one prefers tall tales and inventions to research, that's fine, but don't call it history.

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