Tony Rothman

ELECTRODYNAMICS

Monday 21 Feb. 1977 (1)

SELF-TUTORIAL TAKE-HOME TEST

DUE ~~Friday 25~~ Feb. 1977

Absorption-plus-scattering = "Interception"

1. The pulse of radiation incident on the atom contains $I(\nu)$ ergs of energy per cm^2 of wavefront per unit frequency. The interception crosssection of the atom at frequency ν is $\sigma_{intercept}(\nu)$. Express the total energy ΔE taken out of the pulse by the atom in terms of $I(\nu)$ and $\sigma_{intercept}(\nu)$: $\Delta E =$ ---

2. The atom is idealized by the model of a single electron attached to a position of equilibrium by a spring of such spring constant as to give the electron a natural circular frequency of oscillation $\omega_0 (= 2\pi\nu_0)$. Write here the equation of motion of this electron under the influence of a beam of monochromatic radiation that travels in the z-direction, linearly polarized in the x-direction, neglecting (a) radiation damping force (b) other damping force (both these omissions repaired below) and (c) all displacement of the electron away from the origin $x=0, y=0, z=0$ compared to the very much larger wavelength of the radiation:

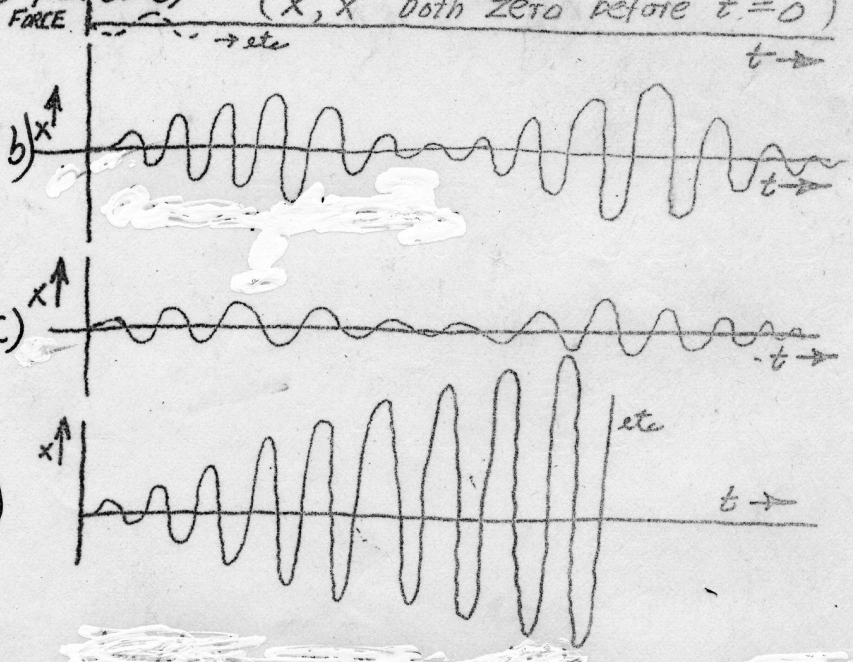
$$m\ddot{x} + m\omega_0^2 x = \begin{cases} -eE_0 \sin\omega t & \text{after } t=0 \\ 0 & \text{before } t=0 \end{cases}$$

3. Which of the curves at the right describes qualitatively ($x, \dot{x}$ both zero before $t=0$)

(a) the displacement x of the electron as a function of the time t when $\omega = \omega_0$ (draw arrow to correct curve)

(b) when $\omega = 1.4\omega_0$

(c) when $\omega = 0.6\omega_0$



ELECTRODYNAMICS Monday 21 Feb 1977 (2)

SELF-TUTORIAL TAKE-HOME TEST DUE Fri ~~Wed~~ (cont'd)

4. The beat phenomenon illustrated at the bottom of the preceding page is given by substituting the right trigonometric functions of time into the following expression — please fill in:

$$x(t) = \frac{-eE_0}{m} \; \frac{\ldots\ldots\ldots\ldots - \frac{\omega}{\omega_0} \ldots\ldots\ldots\ldots}{\omega_0^2 - \omega^2}$$

5. Verify that the expression just given for $x(t)$ reduces to zero at $t=0$ by developing the trigonometric functions in power series and evaluating the first non-zero term; it is ---------

6. Therefore the first non-zero term in the power series for the velocity is .....

Does this too reduce to zero at $t=0$ as it should? --yes---

7. The first non-zero term in the power series for the acceleration $\ddot{x}$ is .....

By reference back to the original eqn. of motion on the preceding page explain why this is or is not a reasonable result, that could have been anticipated from the start:

---

8. Show that your result under item 4 reduces in the limiting case $\omega \to \omega_0$ to the formula

$$x(t) = \frac{eE_0}{2m\omega_0}\left(t\cos\omega_0 t - \frac{1}{\omega_0}\sin\omega_0 t\right)$$

("case of exact resonance")

---

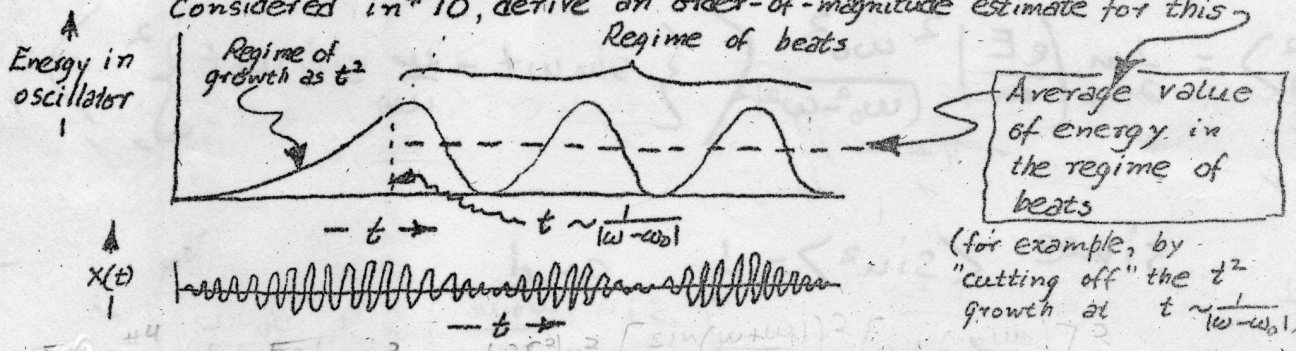
9. For the case of exact resonance, after a long time has gone by, explain why the total energy taken up by the oscillator (still neglecting damping!) is closely proportional to the square of the time, and evaluate the coefficient of proportionality:

$$\begin{pmatrix}\text{energy of}\\ \text{oscillator}\end{pmatrix} \simeq \ldots\ldots K \ldots\ldots t^2$$

SELF-TUTORIAL
TAKE-HOME TEST"

10. When the driving frequency is only a little way off exact resonance, and one limits attentions to times t such that the frequency difference hardly shows up [$(\omega-\omega_0)t$ substantially less than a radian], explain in the sharpest clearest way you can why the growth of (oscillator) energy with time follows the same law as in #9, with the same constant of proportionality, ...
.....
.....

11. Still considering a driving frequency ω that does not differ percentagewise greatly from ω_0, but turning attention to later times than the t from zero to $\frac{\sim 1}{|\omega-\omega_0|}$ as considered in #10, derive an order-of-magnitude estimate for this



$E_{av} \sim$ ..... because ----------

.....

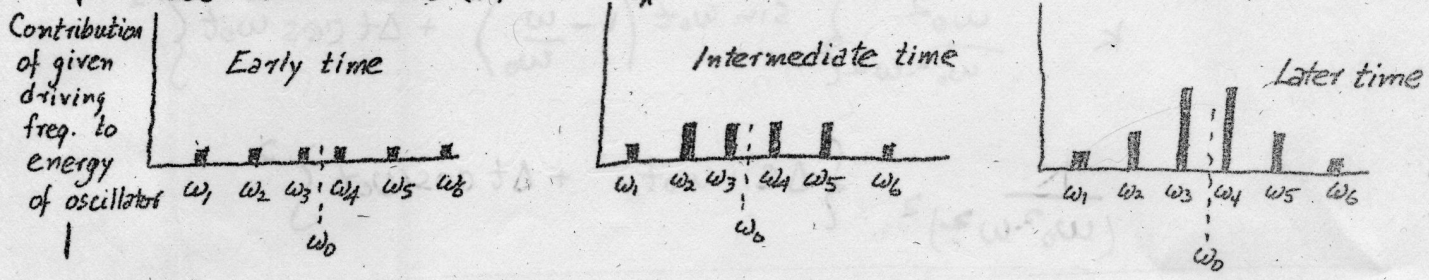
12. The energy incident per cm^2 and per sec in the driving radiation, as averaged over time, is (express in terms of E_0!)
$c\frac{\langle E^2+B^2\rangle_{AV}}{8\pi} =$ .....

13. When instead of one driving frequency, ω, there are many driving frequencies, $\omega_1, \omega_2, \omega_3, \ldots\ldots$, ranged over a whole interval of circular frequencies on either side of ω_0, none of which differs percentagewise greatly from the resonance frequency ω_0, and when we consider the contributions of these several driving frequencies to the energy of the

oscillator to be independent [Why? When one writes $x = x_1(t) + x_2(t) + x_3(t) + \ldots$, where $x_i(t)$ is the response to driving (circular) frequency ω_i, and when one considers the time average value of $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2$ over an interval of time containing many oscillations, why is it reasonable to drop (the contribution of) terms like $\dot{x}_2(t)\dot{x}_3(t)$ and $x_2(t)x_3(t)$ .....

.....] explain (as in (a), (b), .. below) why

these contributions (looking apart from (or averaging over) beat fluctuations, not shown) look as illustrated here:



(a) Why do the contributions of all driving frequencies grow as t^2 at "early times"? .....

.....

(b) Why have contributions of ω_1 and ω_6 stayed at previous low value at "intermediate time" and "later time" .....

.....

(c) At "later time", give (order of magnitude) formula for envelope of chart profile; thus,

$$\left(\begin{array}{l}\text{contribution of driving (circular) frequency } \omega_i \text{ to energy of oscillator of natural circular frequency } \omega_0\end{array}\right) \sim \ldots\ldots\ldots\ldots$$

as function of $|\omega_i - \omega_0|$ for later times; i.e., for $|\omega_i - \omega_0|\,t > 1$

(d) Explain why even for this "later time" the formula for "contribution" has to be changed for any frequency ω_i sufficiently close to ω_0 to (contribution) $\sim$ .....

OVER

Explanation .....

.....

"SELF-TUTORIAL TAKE-HOME TEST"

ELECTRODYNAMICS

Monday 21 Feb '77 (5)

14. The "effective width" $|\omega_i - \omega_0|$ of the "region (see diagram on p.(4)) of resonance" (within which "contribution" goes as t^2) is ~~of the order~~ (express in terms of t)

$$\Delta\omega \sim \text{.....}$$

15. ~~The contribut~~ Let there be $\frac{dN}{d\omega}\,d\omega$ driving frequencies $\omega_6, \omega_7, \omega_8, \ldots$ in the interval of circular frequency $d\omega$. Then the number of driving frequencies within the "effective width" of the "region of resonance" is (express in terms of $dN/d\omega$ and t and any other relevant quantity or quantities)

$$\Delta N \sim \text{.....}$$

16. The contribution of each of the driving frequencies (within the effective width of the region of resonance) to the energy of the oscillator is (from #9)

$$\begin{pmatrix}\text{contribution to energy of oscillator}\end{pmatrix} \simeq \text{.....}\, t^2$$

17. Therefore the total contribution of the whole incident spectrum to the energy of the oscillator (combining 15 and 16) is

$$\Delta E = \begin{pmatrix}\text{energy of oscillator}\end{pmatrix} \simeq \text{.....}$$

18. The radiant energy that flows past the atom in time t, per cm^2 of interception area, ~~(see #12)~~ and per unit interval of circular frequency, is (see #12 and #15):

$$\begin{pmatrix}\text{energy per } cm^2 \text{ per unit interval of } \omega\end{pmatrix} = \text{.....}\cdot t$$

~~19. Defining interception cross-section of the mode~~

19. The radiant energy per cm^2 per unit interval of ordinary frequency, ν, during the time t, is (multiply #18 by 2π? or divide?)

$$I(\nu) = \begin{pmatrix}\text{energy per } cm^2 \text{ per unit interval of } \nu\end{pmatrix} = \text{.....}\cdot t$$

20. When the absorption is all concentrated near a single line of frequency ν_0, explain why it is right to put $\Delta E = \int \sigma_{\text{inter}}(\nu) I(\nu)\, d\nu = I(\nu_0) \int_{\text{over resonance}} \sigma(\nu)\, d\nu$. Explanation .....

21. Taking the energy ΔE imparted to the oscillator out of #17, and the strength $I(\nu_0)$ of the spectrum near resonance from #19, deduce (an approximate value for the) integrated strength of the atomic absorption line ("one oscillator model"):

$$\begin{pmatrix}\text{integrated value of interception cross section}\end{pmatrix} = \int_{\text{over resonance}} \sigma(\nu)\, d\nu = \frac{\Delta E}{I(\nu_0)} \simeq \ldots\ldots\ldots\ldots$$

22. State in the best, sharpest, clearest words you can the qualitative story of how it comes about that an oscillator with a sharp frequency manages to take up energy from a spectrum containing a practically continuous distribution of frequencies (summarizing the "main point" of items #1 through #21)

. . . . . .

. . . . . .

. . . . . .

. . . . . .

. . . . . .

. . . . . .

IMPROVED ~~ANALYSIS~~ DETERMINATION OF INTEGRATED CROSSSECTION FOR OSCILLATOR MODEL OF ATOM (Foregoing analysis could be improved so that instead of order of magnitude approximation we got precise values. However, to have put in all that mathematical detail would have obscured the physics that is going on. Now that that physics is in hand, we turn to the case of pulse-wise excitation to get additional insight and a quick calculation of the integrated cross section.)

23. A pulse of electromagnetic radiation, travelling in the z-direction, and polarized in the x-direction,

passes over the harmonic-oscillator model atom at time $t = 0$ and once again we evaluate the integrated value of the interception cross-section from #21,

$$\int_{\text{over resonance}} \sigma_{\text{interception}}(\nu)\, d\nu = \frac{\Delta E}{I(\nu_0)},$$

thus requiring that we evaluate (a) the energy ΔE imparted to the atom by the pulse and (b) the intensity of the continuous spectrum into which the pulse can be Fourier-analyzed. Do this Fourier analysis first, recollecting the key results about Fourier integrals:

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} E(\omega)\, d\omega$$

("representation of pulse — or "any" disturbance — as a Fourier integral")

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{+i\omega t} E(t)\, dt$$

("inversion of Fourier integral to obtain the Fourier amplitude $E(\omega)$")

$$\int_{-\infty}^{+\infty} |E(t)|^2 dt = \int_{-\infty}^{+\infty} |E(\omega)|^2 d\omega$$

("some total energy (or $4\pi\mathcal{E}$) obtained by integrating over frequency as obtained by integrating over time")

Let the electric field in the pulse be $E(t) = A\,\delta(t)$, where A is a constant. What is the Fourier amplitude, $E(\omega)$, of the electric field?

$E(\omega) =$ value of integral (2) at $t=0$ $= \dots\dots\dots\dots = \frac{A}{\sqrt{2\pi}}$

24. The total amount of energy-per-cm^2 that flows past the atom as any pulse goes by is

$$\left(\begin{matrix}\text{energy} \\ \text{per cm}^2\end{matrix}\right) = \dots\dots\dots \int_{-\infty}^{+\infty} |E(t)|^2 dt$$

↑ insert correct factor

(remember magnetic energy has to be added to electric energy!)

"SELF-TUTORIAL TAKE-HOME TEST"

ELECTRODYNAMICS

Monday 21 Feb '77 (8)

25. Recalling that electric field has the units $\sqrt{\frac{g}{cm\,sec^2}}$, write down here in g, cm, sec the units on left and right sides of #24

$$\underbrace{\left(\begin{matrix}\text{energy}\\ \text{per cm}^2\end{matrix}\right)}_{\text{expressed in g, cm, sec}} = \underbrace{\qquad}_{\text{units of your factor in g, cm, sec}} \cdot \underbrace{\int_{-\infty}^{+\infty} |E(t)|^2\,dt}_{\text{units of this integral in g, cm, sec}}$$

26. Rewrite this result as

$$\left(\begin{matrix}\text{energy}\\ \text{per cm}^2\end{matrix}\right) = \underbrace{\qquad}_{\text{your same factor}} \cdot \int_{-\infty}^{+\infty} |E(\omega)|^2\,d\omega$$

and further rewrite as

$$\left(\begin{matrix}\text{energy}\\ \text{per cm}^2\end{matrix}\right) = \qquad \cdot \int_{0}^{\infty} |E(\omega)|^2\,d\nu$$

("revised factor")

(only half the previous range)

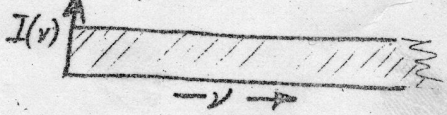
(ordinary frequency)

27. Recognize that what this formula is talking about is $I(\nu)$ and use it to evaluate $I(\nu)$:

$$\left(\begin{matrix}\text{energy per}\\ \text{cm}^2 \text{ per}\\ \text{unit range}\\ \text{of ordinary}\\ \text{frequency,}\\ I(\nu)\end{matrix}\right) = \underbrace{\qquad}_{\text{above "revised factor"}} |E(\omega)|^2$$

28. Evaluate $I(\nu)$ for the δ-function pulse $E(t) = A\,\delta(t)$

$$I(\nu) = \ldots\ldots\ldots\ldots$$



Now turn to the effect of the pulse on the energy of the electron.

"SELF-TUTORIAL TAKE-HOME TEST"

ELECTRODYNAMICS Mon. 21 Feb. 1977 (9)

29. Integrate the equation of motion

$$m\ddot{x} + m\omega_0^2 x = -eE(t) = -eA\delta(t)$$

from $t = 0-\varepsilon$ to $t = 0+\varepsilon$ and go to the limit $\varepsilon \to 0$ and recognize that $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ were all zero before the arrival of the pulse. What result remains after doing the integration and going to the limit?

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30. How much therefore is the energy-uptake of the electron in this "hammer blow" excitation?

$$\Delta E = \text{---}$$

31. And how much do you get for the integrated cross section from the formula

$$\int_{\text{over resonance}} \sigma_{\text{interception}}(\nu)\, d\nu = \frac{\Delta E}{I(\nu_0)} = \frac{\pi e^2}{mc}$$

#31 = #28

32. Explain the discrepancy with the result given in Jackson, chap 17,

$$\int_{\text{over resonance}} \sigma_{\text{interception}}(\omega)\, d\omega = \frac{2\pi^2 e^2}{mc}$$

No discrepancy: this integral is over $d\omega$ gives factor of 2π Good

(END OF THIS "SELF TUTORIAL TAKE-HOME TEST")